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Fractal Analysis of Dynamic Economic Processes in the Field of Investment Decisions on the Warsaw Stock Exchange

Analiza fraktalna dynamicznych procesów gospodarczych w zakresie decyzji inwestycyjnych na Giełdzie Papierów Wartościowych w Warszawie

Abstract

The main aim of this paper is to identify and evaluate the use of nonlinear methods of analysis of dynamic economic processes represented by one-dimensional time series and observe the state of their behavior to study the investment attractiveness of joint-stock companies on the Warsaw Stock Exchange. The study uses chaos theory as one of the novel methods of describing capital markets, which is an advantage over the classic methods of capital market analysis and describes how it functions as a linear system. The fractal analysis of selected companies listed on the Warsaw Stock Exchange detected the existence of long-term memory. Joint-stock companies on the Polish stock exchange have positive Lyapunov exponents, fractal dimensions in the form of fractional numbers, and therefore, the possibility of examining deterministic chaos using tools. The price charts of companies' shares indicate that these are not straight lines but lines that form a zig-zag shape, characteristic for nonlinear systems. Research shows that the Polish capital market has fractal properties, and thus analysis using deterministic chaos methods is possible.

Keywords: Deterministic Chaos, Capital Markets, Warsaw Stock Exchange.
JEL: G17

Streszczenie

Głównym celem artykułu jest identyfikacja i ocena zastosowania metod analizy nieliniowej dynamicznych procesów gospodarczych reprezentowanych przez jednowymiarowe szeregi czasowe oraz obserwacja stanu ich zachowania do badania atrakcyjności inwestycyjnej spółek akcyjnych na Giełdzie Papierów Wartościowych w Warszawie. W badaniu wykorzystano teorię chaosu jako jedną z nowatorskich metod opisu rynków kapitałowych, co stanowi przewagę nad klasycznymi metodami analizy rynku kapitałowego opisującego jego funkcjonowanie jako układu liniowego. Przeprowadzona analiza fraktalna wybranych spółek notowanych na Giełdzie Papierów Wartościowych w Warszawie wykryła występowanie pamięci długofalowej. Spółki akcyjne z polskiej giełdy posiadają dodatnie wykładniki Lapunowa, wymiary fraktalne w postaci liczb ułamkowych, a w związku z tym, możliwość badania za pomocą narzędzi chaosu deterministycznego. Wykresy cen akcji spółek wskazują, że nie są to linie proste, a linie zygzakowate, charakterystyczne dla systemów nieliniowych. Badania pokazują, że polski rynek kapitałowy posiada właściwości fraktalne i istnieje możliwość analizy z wykorzystaniem deterministycznych metod chaosu.

Słowa kluczowe: chaos deterministyczny, rynek kapitałowy, Giełda Papierów Wartościowych w Warszawie.
JEL: G17

1. Introduction

Achieving success when investing in the capital market is extremely complicated, as the capital market is very dynamic and is almost never in equilibrium. From a methodological point of view, the strategies for operating in the capital market are the most difficult of all the strategies used in the economy. Investors look for justification for their investment decisions, using fundamental, technical or other portfolio analysis, which never completely assure success (Siemieniuk, 2001). In this situation, the theory of deterministic chaos becomes very attractive as it offers the technique of time series immersion and the reconstruction of the behavior of entire multidimensional processes through the dynamics of a single time series of share prices. This is possible in theory, as stated in the Takens (1981) statement, where the observation time delays make it possible to obtain a good approximation of the original state space and dynamics of the system based only on observation of one of its variables (Nowiński, 2007).

The lack of discussion in the literature about the use of the chaos theory to study the investment attractiveness of listed companies was the reason for undertaking research on companies listed on the Warsaw Stock Exchange (GPW), supported by the theory of deterministic chaos. This article is an attempt at a fractal analysis of the Polish capital market, using examples of selected companies listed on the GPW, in terms of investment attractiveness.

The aim of this article is to identify and evaluate the use of nonlinear analysis methods of dynamic economic processes represented by one-dimensional time series and observe the state of their behavior to study the investment attractiveness of joint-stock companies on the GPW.

Taking into account the purpose of the publication, the main hypothesis formulated was that the investment attractiveness of joint-stock companies on the Polish stock exchange can be forecast. This means that the future behavior of a given company on the GPW can be predicted using the theory of deterministic chaos. In order to verify the main hypothesis adopted in this article, several measures resulting from the theory of deterministic chaos were used, such as Lyapunov exponents, fractal dimension, correlation dimension and Hurst exponents.

This article contains a short introduction describing nonlinear analysis, followed by a literature review in terms of the use of chaos theory with financial time series, the essence of deterministic chaos. The next section of the paper describes the research methodology, followed by the results and discussion. The last section contains discuss conclusions and proposes future research directions.

2. Literature Review

The creators of the chaos theory, also called the complexity theory or the theory of complex systems, were mathematicians and physicists. The modern chaos theory has been applied in programming, engineering, technical sciences, microbiology, finance, meteorology, philosophy, politics, demographic research and psychology (Jaditz and Sayers, 1993; Peters, 1991; Sneyers, 1997; Gurgul and Suder, 2012; Mosdorf, 1997). An important element determining the chaotic nature of the system is its susceptibility to initial conditions (Tempczyk, 2002). These conditions are related to the nonlinear differential equations that underlie the chaos theory. In a nonlinear dynamic system, the system's state depends on transformations occurring in time (Jordan and Smith, 2007). An example of sensitivity to initial conditions is the *butterfly effect*. This theory was formulated Edward Lorenzwho, who concluded that local pressure changes caused by the movement of a butterfly's wings in Brazil could trigger a hurricane in Texas. In reference to the economy, an example of the *butterfly effect* may be the financial crisis in the United States in 2008, which caused huge drops in the capital markets of Europe and Asia. Edward Lorenz became a pioneer of the chaos theory (Stewart, 1994). Researchers use the chaos theory to describe nonlinear systems occurring in nature, especially in the immediate human environment (Gurgul and Suder, 2013). When it comes to social sciences, it can be applied in forecasting models in the economy (e.g., to describe securities exchange rates), to describe people's behavior, especially crowd psychology, and to explain traffic jams and other economic and social occurrences (Gurgul and Suder, 2013). As Tebyaniyan et al. report (Tebyaniyan et al., 2020), "studying and recognizing the behavior of returns on securities has always been an interest of investors and researchers since the dawn of capital markets. Chaos theory and fractal analysis are the novel theories in this area". In the literature (Vlad et al., 2011) research discusses the main ideas of chaos theory and the meaning of nonlinearities in mathematical models. In the publication, Vlad et al. (2011) write that chaos and order are apparently two opposing terms. Even more surprising is the fact that some exact symmetry can be found in chaos (Feigenbaum numbers). Another publication analyses the world financial crisis from the perspective of chaos theory based on the most important global stock indexes, such as the Dow Jones Index, London FTSE 100 Index, France CAC40 Index, Nikkei 225 Index, Hong Kong Hang Seng Index and Shanghai Composite Index (Xiangqing and Meilin, 2009). Chaos theory can be an effective tool in capital market modelling (Xu et al., 2008). Since the end of the 1980s, in the literature there has been a discussion about whether there is chaos in economic time series (based on nonlinear dynamics) that can be used in research on how economic processes function (Peters, 1997; Bullard and Butler, 1993; Nazarko et al., 1999; Orzeszko, 2005; Nowiński, 2007; Siemieniuk, 2001; Chorafas, 1994; Levy and Sarnat, 1990; Dockner et al., 1997; Łobos et al., 2017; Buła, 2013, 2017, 2019). Using a nonlinear dynamic system with a discrete delay, attempts were made to model the evolution over time of the stock exchange price index and net savings resources in investment funds on foreign markets (Dobrescu et al., 2013). The literature has not provided solid support for chaos because of the high level of noise that occurs in most economic time series, the relatively small size of data samples, and the poor resilience of chaos tests for this data (Faggini and Parziale, 2016). For decades, chaos theory has been one of the hottest topics in science, but so far has not been sufficiently investigated in financial theory or in practice (Klioutchnikov et al., 2017). A significant problem related to the study of financial time series from a fractal perspective is the fact that their graphical representations should be classified as natural stochastic fractals. Due to the inability to determine *a priori* the laws governing the fluctuations of the studied quantities, it is also impossible to calculate the fractal dimension, and it is necessary to use appropriate estimation methods (Buła, 2017, p. 14). The method based on the theory of chaos not only uses the prediction ability resulting from periodic density in chaotic dynamical systems, but also avoids the problem that a single predicted value is not accurate due to the sensitivity of the initial value (Chen et al., 2021).

In chaos theory, using the time series immersion technique and the reconstruction of the behavior of entire multidimensional processes through the dynamics of a single time series of share prices, it is possible to obtain a good approximation of the original state space and dynamics of the system using observation time delays only based on observation of one of its variables (Takens, 1981).

For a very high level of noise, the application of Takens' theorem is impossible in practice. This may prevent the correct reconstruction of the system dynamics form and may force the researcher to use a modification of the theorem. However, Takens' embedding theorem provides a theoretical basis for the reconstruction of the dynamics of the system.

An embedding dimension is associated with a time series immersion, which is defined as successive series of time-delayed series points, which are treated as points in the m-dimensional embedding space, where m is the embedding dimension of the time series of the dynamical system. Another parameter related to the time series immersion method is the time delay defined as the percentage of the orbit for each embedding dimension m. When analyzing the literature on this subject, there is no unambiguous method for the optimal selection of the delay value *t* and the embedding dimension m, and there is no one optimal set of parameters. If it is too small, the embedded vectors will be very close, which will cause redundancy. If the delay is too large, the embedded vectors will not correlate at all, irrelevance will occur, and the reconstructed attractor will fill almost the entire state space (Nowiński, 2007).

3. Research Methodology

The research in the paper was conducted on time series of IT companies listed on the GPW. Five companies operating on the GPW and five that existed until they were declared bankrupt and ceased trading were selected for the study. The companies operating on the Polish stock exchange at the time of the research which were selected for the analysis were Comp S.A. (CMP), NTT System SA (NTT), SIMPLE SA (SME) and Dębica, PKOBP. Among the companies that declared bankruptcy and ceased to operate on the Polish stock exchange, the companies that qualified for research were Swarzędz, Techmex, Zntklapy, Uniwersal and Próchnik. The work includes an analysis of a number of share quotations of companies such as Dębica, Próchnik, Swarzędz and Universal since 1995,[1] PKOBP, Techmex since 2004 and

[1] From this year, quotations took place every day and created complete time series.

Zntklapy since 2007, i.e. from the moment the company was listed on the stock exchange. For bankrupt companies, the research was completed in the year in which the court declared the company bankrupt on the Polish stock exchange, i.e. for Universal, the research was completed in 1999, for Tonsil in 2005, for Zntklapy in 2010, and for Swarzędz and Techmex in 2011, and for Próchnik in 2019. Among the IT companies listed on the Polish stock exchange, the research covered the period April 12, 2007 to April 6, 2021. The closing price was selected for all companies when constructing the time series of stock quotes. Joint-stock companies with a longer history of operation on the Polish stock exchange were selected for the study, because this meant longer time series covered by the study and, therefore, more accurate research results. In the empirical research on the stock quotes of companies on the GPW, data was collected from the *bossa.pl* website. The research results were calculated and presented using the Recs computer program, which implements the algorithms of deterministic chaos methods. The program interface is based on WPF (Windows Presentation Forms) technology. This software enables the fractal analysis of any time series.

For the collected data to be analyzed using methods connected to deterministic chaos theory, the trend lines were eliminated. This operation can be performed using several methods. In the first method, a times series of stock returns is constructed according to the following formula (Weron and Weron, 1998):

$$S_i^{\tau} = \frac{P_i - P_{i-\tau}}{P_{i-\tau}} \tag{1}$$

where:

P_i – stock price on a given day;

τ – time interval in days.

The next method uses a Fourier analysis of the power spectrum and removal of the first frequencies for the tested series. As already shown in the literature (Siemieniuk, 2001), for stock exchange data from the Polish stock exchange, a more appropriate method of preparing experimental data is to construct a series of stock price returns. Therefore, assuming a time delay of $\tau = 7$, a series of stock price returns selected for the analysis of companies on the Polish stock exchange were constructed, in accordance with the above formula. For the IT companies Comp S.A. (CMP), NTT System SA (NTT) and SIMPLE SA (SME), for which the study covered the four-year period of 2007–2011, a time delay τ equal to 1 was assumed. This was dictated by short time series. This selected time delay value means that the effects of short-term memory may be eliminated. This may be evidenced among others by a trend occurring in a time series.

The elimination of the trend in the examined time series of stock prices makes it possible to eliminate the effects of signals coming from the market, e.g., changes in decisions made by investors regarding the purchase or sale of shares, guided in one case by the hope of increases in share prices, and on the other hand by desperation related to price drops and the imminent bankruptcy of a given company on the stock market.

The next step in fractal analysis involves analyzing the fractal dimension level, the Hurst exponent, and the Lapunov exponent. The Lyapunov exponent measures the attractor dynamics and measures the system's susceptibility to changes in initial conditions, i.e., how the prediction based on an inaccurate estimate of the initial conditions will deviate from the actual development of a given system. The Lyapunov exponent shows the rate at which the ability to predict the system's future behavior is lost. In the case of chaotic systems, the Lapunov exponent is a positive number (Siemieniuk, 2001).

The analysis begins with the determination of the time delay. For this purpose, the autocorrelation function is calculated with a specific relationship (Siemieniuk, 2001):

$$C(\tau) = \frac{1}{N} \sum_{i=0}^{N-\tau-1} x_i \cdot x_{i+\tau} \tag{2}$$

where:

N – denotes the sample size;

t – time delay;

x_i – the value of the i-th sample.

It is impossible to determine all Lyapunov coefficients for measurement data. However, it is possible to determine the value of the largest Lyapunow exponent. In this case, two points separated by at least one orbital period are selected on an attractor immersed in D-dimensional space. The distance between these points is L(τj). Then the distance of the selected points is calculated after a certain period of evolution. The new distance of the pair of points is L'(τj+1). The largest Lyapunov exponent is calculated according to the formula (Peters, 1997, p. 156).

$$L_1 = \frac{1}{\tau} \sum_{j=1}^{m} \log_2 \frac{L'(\tau_{j+1})}{L(\tau_j)} \tag{3}$$

where:

m – the capacitive dimension;

τ – the time delay.

Determining the largest Lyapunov exponent is possible when attractor characteristics such as fractal dimension, mean orbital time and time delay are known. The Lyapunov exponent shows whether the behavior of a dynamical system is chaotic. It can be defined in the form of an equation (Schuster, 1993, p. 35).

$$\lambda(z_o) = \lim_{N\to\infty} \lim_{\varepsilon\to 0} \frac{1}{N} \ln \left| \frac{f^N(z_o + \tau) - f^N(z_o)}{\tau} \right| \tag{4}$$

where:

z_0 – initial condition;

f' – derivative of the mapping determining the behavior of the system.

Based on the autocorrelation function, the time delay can be determined, and this is suitable for the reconstruction of the attractor of the examined system (Siemieniuk, 2001). This delay should be large enough to eliminate the short-term memory of the system, but small enough not to lose the cause-and-effect relationships between the information carried by subsequent samples.

The next step of the contained fractal analysis was to determine the fractal dimension of the reconstructed attractor. For a time series of stock quotes of select companies, correlation dimensions were calculated with a time delay of $t = 7$ using the Grassberger-Procaccia method (Grassberger and Procaccia, 1983).

For the experimental data, the correlation dimension *n2*, based on the algorithm proposed by Grassberger-Procaccia, is determined by the following relationship (Schuster, 1993):

$$n_2 = \lim_{l \to 0} \frac{1}{\ln l} \ln \sum p_i^2 \tag{5}$$

where:

$$\sum_i p_i^2 \approx \lim_{N \to \infty} \frac{1}{N^2} \sum_{i,j}{}^{\Theta} \left(l - \left| x_i - x_j \right| \right) = C_2 \left(l \right) \tag{6}$$

where:

Θ – the Heaviside function determining the number of pairs of attractor points (*xi, xj*), the distance of which is less than *l*;

N – the size of the studied set;

L – the distance between points *xi, xj* of the attractor;

C_2 – correlation integral.

The quantity C_2, called the correlation integral, shows the probability of finding a pair of attractor points at a distance shorter than *l*. As the value of *l* increases, C_2 should increase at the rate of l^{n2}.

One of the basic methods for distinguishing random series from non-random ones is analysis of the scaled R/S range, in which the Hurst exponent is determined. It is used to determine long-term memory effects and to detect fractional Brownian[2] motion. Depending on the size of the Hurst exponent, it is possible to say that there is chaos in the system under study.

The Hurst exponent is determined as follows. The measurement data (in the form of sequential samples) are divided into intervals with a fixed number of points equal to N. For each of the intervals, we define a number of characters (Peters, 1997, p. 65):

[2] Brownian motion, also called *white noise*, is characteristic of time series with a Hurst coefficient equal to 0.5, that is, series with a random walk. Successive values in such a series are independent of each other and there is no correlation between them (Peters, 1997, p. 242).

$$T_i = \sum_{j=1}^{i}(x_j - \bar{x}_N) = \sum_{j=1}^{i} x_j - i \cdot \bar{x}_N \tag{7}$$

where:

Ti – denotes the cumulative deviation in N samples;

$\underline{X}$j – sample value at time j;

$\bar{x}_N$ – the arithmetic mean of the data for N samples.

For each of the intervals of the above series, we calculate the quantity R, called the *range*:

$$R = \max(T_i) - \min(T_i) \tag{8}$$

and the standard deviation S.

where:

R – the *range* of the series;

max(*T)* – the maximum value of the series;

min(*T*) – the minimum value of the series.

For each interval, the R *range* is divided by the standard deviation of the original observations S. Then, the average value characteristic of all intervals is determined. The *range* scaled in this way increases as N increases. The Hurst exponent is determined from the relationship (Nazarko et al., 1999).

$$R/S = (a \cdot N)^H \tag{9}$$

where:

R/S – the rescaled *range*;

N– the number of observations;

a – a constant;

H – the Hurst exponent.

There are three size classes for the Hurst exponent (Nowiński, 2007):

- H = 0.5 – the time series is random;
- 0 ≤ H < 0.5 – the time series is antipersistent (returning to average);
- 0.5 < H < 1 – the time series is persistent, chaotic (strengthening the trend).

Capital markets are persistent series, and they are fractals. As H increases, the line becomes smoother, and there is less noise in the system. A high Hurst exponent means lower risk involved in investing in given stock. The Hurst exponent is a measure of the complexity of a fractal series.

4. Results and Discussion

An analysis of the literature on the subject (Peters, 1997; Siemieniuk, 2001; Mosdorf, 1997) shows that one of the methods of deterministic chaos theory is Lapunov exponents, which measure the susceptibility of dynamic systems to a change in initial conditions. The positive Lyapunov exponent determines how a forecast based on an inaccurate baseline estimate will deviate from the true development of the system. In the second approach, the Lapunov exponent makes it possible to determine the time after which the value of the forecasts declines. The negative Lapunov exponent is a measure of phase space point convergence, i.e., it specifies how long the system needs to return to its original state after a disruption. Determining the value of Lapunov's exponents is a strict criterion of chaos. A system is chaotic if it has at least one positive Lyapunov exponent. In such a situation, in the phase space, closely lying trajectories may, after some time, move away from each other at will. Although we can accurately predict the system's behavior for ideally and precisely set initial parameters, in practice, even when the initial conditions are known with finite accuracy, the system becomes unpredictable in a short time.

The results of the empirical research described in Table 1 show that the joint-stock companies on the GPW that have been analyzed have positive Lapunov exponents, i.e., they are chaotic systems, which means that it is possible to analyze them using deterministic chaos tools.

Table 1.

The Lapunov exponent for select joint-stock companies on the GPW

Joint-stock company	Value of the Lapunov exponent	Cycle in months 1 / exponent L Information decay period
Comp S.A. (CMP)	0.012	(1/0.012) * (1/30) = 2.78
Dębica	0.027	(1/0.027) * 7/30 = 8.6
NTT System S.A. (NTT)	0.031	(1/0.031) * (1/30) = 1.08
PKOBP	0.024	(1/0.024) * 7/30 = 9.7
Próchnik	0.030	(1/0.030) * 7/30 = 7.7
SIMPLE S.A. (SME)	0.034	(1/0.034) * (1/30) = 0.98
Swarzędz	0.095	(1/0.095) * 7/30 = 2.4
Techmex	0.056	(1/0.056) * 7/30 = 4.1
Universal	0.079	(1/0.079) * 7/30 = 2.9
Zntklapy	0.101	(1/0.101) * 7/30 = 2.3

Source: Author's own research based on bossa.pl.

Table 2 shows the values of the fractal dimension and the correlation dimension obtained for select joint-stock companies on the GPW. Figures 1–3 show fractal dimensions in relation to the time series of stock quotes of select joint-stock companies on the GPW.

Table 2.

The fractal dimension and the correlation dimension for select joint-stock companies on the GPW

Company	Fractal dimension	Correlation dimension (Number of variables)
NTT System S.A. (NTT)	1.4	2
SIMPLE S.A. (SME)	1.4	2
PKOBP	3.1	4
Dębica	4.3	5
Comp S.A. (CMP)	4.7	5
Zntklapy	6.5	7
Techmex	6.9	7
Swarzędz	7.3	8
Universal	8.1	9
Próchnik	8.5	9

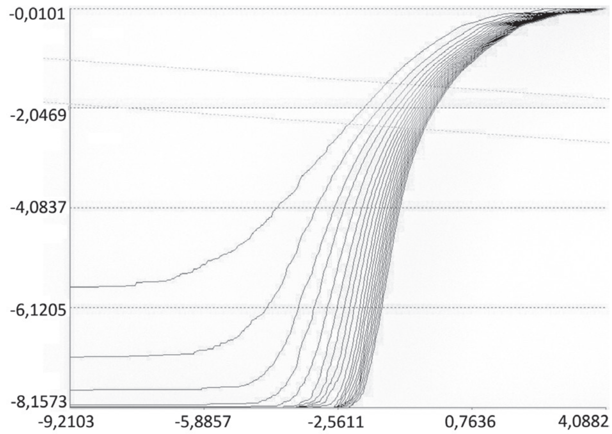
Source: Author's own research based on bossa.pl.

For a time series of stock quotes for NTT, the fractal dimensions were calculated in relation to the stock quote for the period 12.04.2007–06.04.2021 for embedding 1–20 and the time delay $\tau = 1$. The results of the analysis are presented in Figure 1. Correlation dimensions stabilize from the fifteenth embedding dimension, assuming the value of 1.4. Since it is a non-integer value, deterministic chaos is observed in the series. This proves the existence of two independent variables in the analysis process. The data in Table 2 clearly show the fractal nature of the time series of stock quotes for NTT. The fractal dimension of 1.4 means that it is possible to model this market with two variables.

Figure 1.

Determining the fractal dimension based on the correlation dimensions for the stock quotes of NTT for the period 12.04.2007 – 06.04.2021, τ = 1:

a) lnC (l) curves

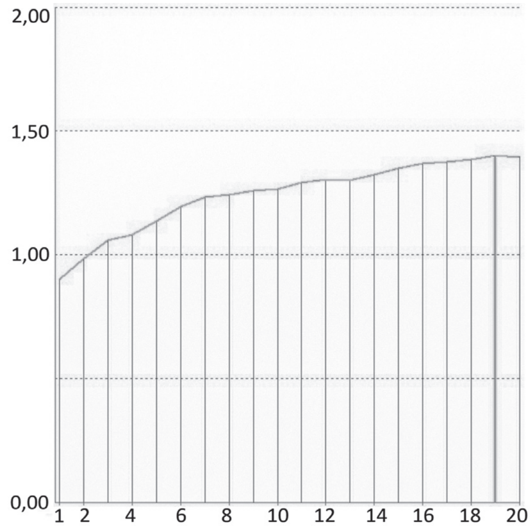


X axis – estimate of the function ln(l);

Y axis – graphs of $lnC_2(l)$ curves as a function of ln(l).

Source: Own research based on bossa.pl.

b) Correlation dimensions for dips 1–20 calculated based on the slopes of tangent regression lines to the curves from the marked image



X axis – d (denotes embedding dimension);

Y axis – n_2 (denotes correlation dimension).

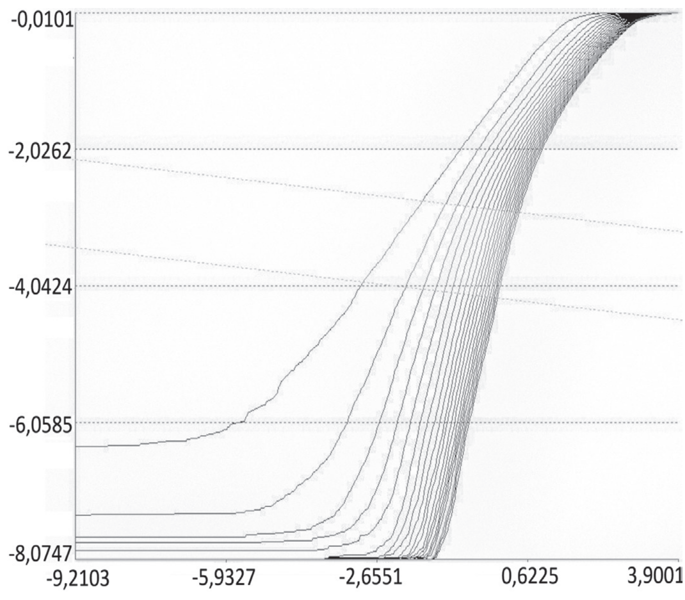
Source: Own research based on bossa.pl.

For a time series of stock quotes for the company SME, the fractal dimensions were calculated in relation to the stock quote for the period 12.04.2007 – 06.04.2021 for embedding 1–20 and the time delay τ = 1. The results of the analysis are presented in Figure 2. Correlation dimensions stabilize from the fourteenth embedding dimension, assuming the value of 1.4. Since it is a non-integer value, deterministic chaos is observed in the series. This proves the existence of two independent variables in the analyzed process. The data in Table 2 clearly show the fractal nature of the time series of stock quotes of SME. The fractal dimension of 1.4 means that it is possible to model this market with two variables.

Figure 2.

Determining the fractal dimension based on correlation dimensions for the stock quotes of SME for the period 12.04.2007 – 06.04.2021, τ = 1:

a) lnC (l) curves

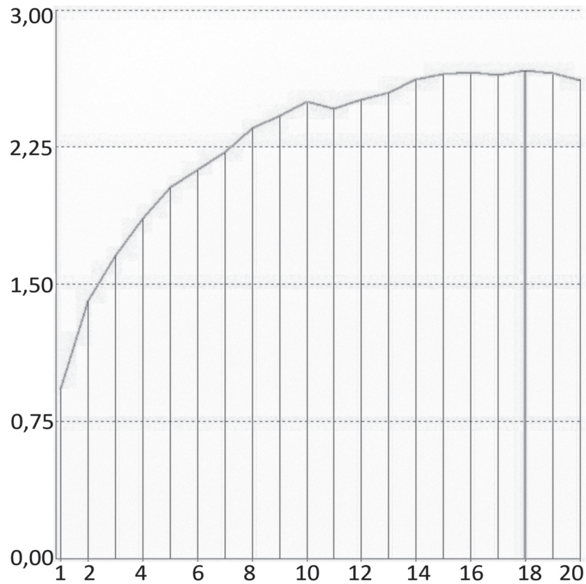


X axis – estimate of the function ln(l);

Y axis – graphs of $\ln C_2(l)$ curves as a function of ln(l).

Source: Own research based on bossa.pl.

b) Correlation dimensions for embedding 1–20 calculated based on the slopes of tangent regression lines to the curves from the marked image



X axis – d (denotes embedding dimension);

Y axis – n_2 (denotes correlation dimension).

Source: Author's own research based on bossa.pl.

The research conducted showed that the joint-stock companies on the Polish stock exchange that were analyzed have fractal dimensions in the form of fractional numbers (Tab. 2), i.e. they are chaotic systems, which means that it is possible to analyze them using deterministic chaos tools. The fractal dimension is important information about the system because it makes it possible to calculate the minimum number of dynamic variables needed to describe the functioning of a given listed company. The lower the number, the less complicated and more predictable its functioning on the stock market. Analysis of the data contained in Table 2 shows that companies in good economic condition have a lower correlation dimension, and thus a lower number of dynamic variables necessary to analyze a given company on the stock exchange, compared to companies that went bankrupt on the GPW. R. Buła's research has shown that instruments with a higher fractal dimension are assigned a higher level of risk, while this is only true when the length of the investment horizon decreases to zero (Buła, 2015, p. 465). In turn, the authors of the publication demonstrate that the validity of the forecast is estimated using Lyapunov exponents. The Lyapunov exponent makes it possible to calculate the time after which the forecasts cease to be of value. Although for perfectly and precisely set initial parameters the behavior of the system can be predicted accurately, in practice, even when the initial conditions are known with finite accuracy, the system quickly becomes unpredictable.

Table 3 and Figures 3–6 compare the Hurst exponent for select listed companies in Poland. Table 3 shows that due to the R/S analysis of select joint-stock companies, the value of the Hurst exponent indicates the occurrence of deterministic chaos.

Table 3.

Hurst exponent for select joint-stock companies on the GPW

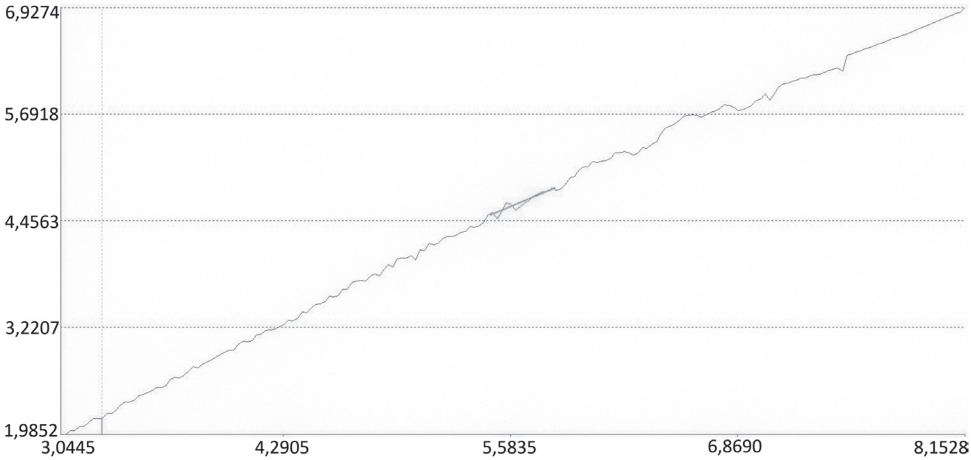
Joint-stock company	Value of the Hurst exponent
NTT System S.A. (NTT)	0.89
SIMPLE S.A. (SME)	0.88
Comp S.A. (CMP)	0.87
PKOBP	0.77
Dębica	0.71
Swarzędz	0.61
Zntklapy	0.59
Techmex	0.59
Universal	0.56
Próchnik	0.56

Source: Author's own research based on bossa.pl.

Analysis of the data in table 3shows that the Hurst exponent for NTT is 0.89. The characteristics of deterministic chaos for NTT were calculated for the timeframe 12.04.2007 – 06.04.2021. Figure 3 shows the graph and the size of the Hurst exponent for the shares of NTT. Figure 4 shows the stock chart of NTT on the GPW for the period 11.2020 – 09.2021. The results of the analysis of the Lapunov exponent included in table 1 show that for NTT the system loses information after approximately 1.8 months, which is the period for which the forecast is valid. After this time, the company's shares analysis should be repeated using deterministic chaos methods. The above conclusions are confirmed by the study of the NTT stock quote chart (Fig. 4) in 2021, showing that the stock quote of NTT with an exponent of H equal to 0.89 turned out to be low risk for stock market investors. R/S analysis reveals that NTT is in good economic condition and is still operating on the Polish stock exchange.

Figure 3.

R/S analysis results for the time series of the increase in stock quotes of NTT for the period 12.04.2007 – 06.04.2021



X axis – estimate of the function ln (N);

Y axis – logarithm of the ln(R/S) function of the scaled range.

Source: Author's own research based on bossa.pl.

Figure 4.

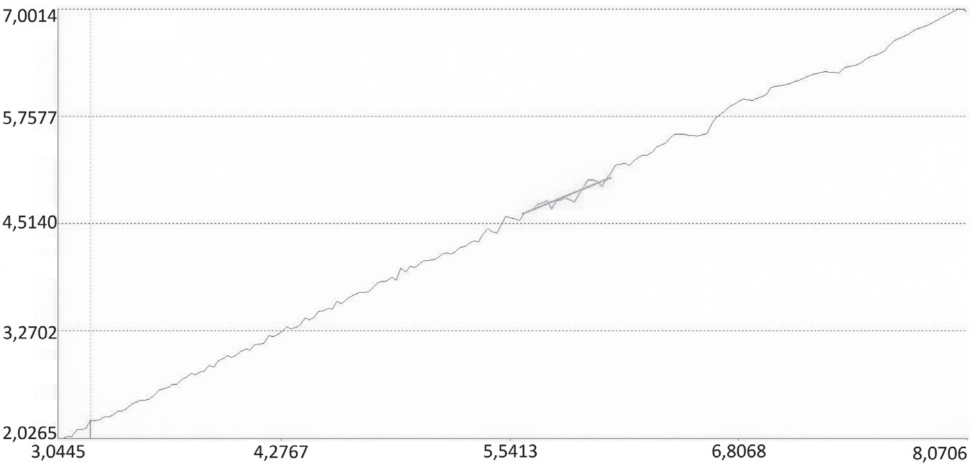
NTT stock chart for 2020–2021

Source: https://www.bankier.pl/inwestowanie/profile/quote.html?symbol=NTTSYSTEM (accessed on 07.09.2021).

Analysis of the data in Table 3 shows that the Hurst exponent for SME is 0.88. Calculations concerning the characteristics of deterministic chaos for SME were made for the period 12.04.2007 – 06.04.2021. Figure 5 shows the graph and the size of the Hurst exponent for the shares of SME. Figure 6 shows the stock chart of SME on the GPW for the period 11.2020 – 09.2021. The analysis of the Lapunov exponent included in Table 1 shows that for SME the system loses information after approximately 0.98 months, which is the period for which the forecast is valid. After this time, the company's shares analysis should be repeated using deterministic chaos methods. The above conclusions are confirmed by the analysis of the SME share price chart (Fig. 6) in 2021, showing that the stock quote of SME with an exponent H equal to 0.88 turned out to be low risk for stock market investors. The R/S analysis shows that SME is in good economic condition and is still operating on the GPW.

Figure 5.

R/S analysis results for the time series of the increase in stock quotes of SME for the period 12.04.2007 – 06.04.2021



X axis – estimate of the function ln (N);

Y axis – logarithm of the ln(R/S) function of the scaled range.

Source: Author's own research based on bossa.pl.

Figure 6.
SME stock chart for 2020–2021

Source: https://www.bankier.pl/inwestowanie/profile/quote.html?symbol=SIMPLE (accessed on 07.09.2021).

The research included in the publication showed that the joint-stock companies on the Polish stock exchange that were analyzed have Hurst exponents higher than 0.5 (Tab. 3), which means that they are chaotic systems. This in turn means that it is possible to analyze them using deterministic chaos methods. If the Hurst exponent (H) is lower, the series is more random. A higher value of the Hurst exponent (H) means that investing in a given security is less risky because there is less *noise* in that series. This relationship was analyzed using the example of selected joint-stock companies on the Polish stock exchange. The data in Table 3 shows that the above relationship has been positively verified, i.e. companies in good economic condition have a higher Hurst exponent compared to companies that went bankrupt on the GPW.

However, a greater risk of sudden changes is associated with stocks with a high Hurst exponent (Peters, 1997). The susceptibility of the system to changes in initial conditions is indicated by the Lyapunov exponent. It makes it possible to determine how forecasts are based on an imprecise estimate of baseline conditions and may differ from the true development of a given system. It also shows the rate of decline in the ability to predict future behavior. Therefore, it is important to study the forecast validity using Lyapunov exponents carefully.

5. Conclusions

According to the findings, the behavior of a joint-stock company on the GPW can be projected in terms of its attractiveness in the short term, estimated using the Lapunov exponent, which determines the time after which information is lost and predictability fades. After this period, using fractal analysis, the calculations should be repeated. Following the forecast decay period (calculated using the Lyapunov exponent), the method should be applied with caution, as there is a risk of sudden changes when the limit is exceeded, after which the forecasts disappear; the interpretation of the results may then cause problems. The fractal dimension makes it possible to determine the minimum number of dynamic variables needed to describe the functioning of a given listed company. The lower the number, the less complicated and more predictable its functioning on the stock market. As research has shown, the listings of stock exchange companies in Poland with lower investment attractiveness have a higher fractal dimension compared to companies with higher share investment attractiveness. As the Hurst exponent increases, the line becomes smoother, and there is less noise in the system. A high Hurst exponent means that there is lower risk involved in investing in given stock. Research shows that the listings of stock exchange companies in Poland with lower investment attractiveness have a lower Hurst exponent compared to companies with higher share investment attractiveness. Comprehensive analysis of all indicators allows conclusions to be drawn regarding investing in shares of a given company. The value of the indicator can determine whether a listed company is more or less attractive to a stock market investor.

As the research described in this article shows, the Polish capital market has fractal properties, and thus, it can be analyzed using the methods of deterministic chaos. The results obtained in this article show that fractal analysis of the functioning of joint-stock companies on the GPW is a significant supplement to the classic methods of stock market analysis. The fractal analysis does not bring us closer to forecasting the value of shares on a specific day, but it makes it possible to assess the probability of certain market behavior and modeling alternative scenarios of its behavior. The issues presented in this article show that chaos theory can play a major role in the analysis of the stock market and in investor decision-making. It is possible to assess the risk of investing in specific shares in terms of the investment attractiveness of a given company on the GPW. Assessment of investment attractiveness of companies listed on small stock exchanges such as the GPW, using models based on the theory of deterministic chaos, requires an individual approach to each of the entities listed on the stock exchange and to the economic situation and specificity of a given local financial market. The use of the deterministic chaos theory presented in this article does not exhaust all the possibilities in this respect. As the considerations are forward-looking, clearly they are also debatable. The use of chaos methods as an aid for decision-making when investing in small stock exchanges is limited by investors' ability to use the software. For this reason, measures resulting from the chaos theory may, in practice, be periodically estimated and published by stock market analysts.

In the publication, the authors point out that the research hypothesis was proved correct for a specific stock exchange – the GPW, and thus it is not universal in nature, but applies to that stock exchange. It takes into account the specificity of a small stock exchange, such as the GPW, and the limitations resulting from the relatively short time series and changes in the economic environment both in the real economy and in the financial sector in the Polish economy during the period covered by the study. Also, the general conclusion from the verification of the research hypothesis – that the deterministic chaos theory can be used for assessing and projecting the economic condition of joint-stock companies – can be applied to companies listed on other local stock exchanges. At the same time, the interpretation of the coefficients (Lyapunov, Hurst, fractal and correlation dimensions) should take into account the conditions of specific economies, political and economic systems and the local stock exchanges operating in them. The precision of estimates in the case of deterministic chaos theory methods is important, because even a small change in the initial state produces millions of results, which is one of the disadvantages of the approach used. After the forecast decay period (calculated using the Lyapunov exponent), the method should also be applied with caution; there is a risk of sudden changes when the forecast decay period ends; the interpretation of the results may then cause problems. As E. E. Peters writes, classical methods tend to perceive economic systems as balanced systems (point attractors) or as systems oscillating cyclically around a state of equilibrium (limit cycle). However, none of these approaches has been empirically confirmed. It appears that economic time series are indefinite cycles (that is, they have no specific length or time scale). Such cycles occur in nonlinear dynamic systems. Therefore, the theory of deterministic chaos makes it possible to study economic tendencies in terms of their non-linearity.

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